Markov chains 3

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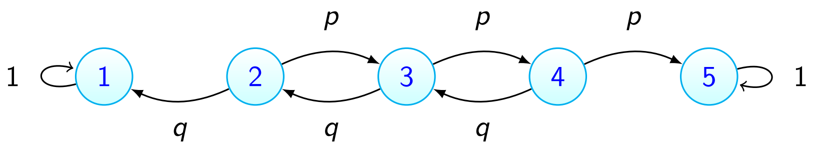
Random walk with 5 positions



$$\mathcal{P} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ q & 0 & p & 0 & 0 \\ 0 & q & 0 & p & 0 \\ 0 & 0 & q & 0 & p \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

$$p + q = 1$$

Example 2

Gambler's ruin

- A gambler wins or loses 1 bitcoin with probabilities p and q respectively; $p + q = 1$.
- Results of successive bets are independent.
- Initial fortune: k bitcoins.
- Stopping rule: When ruined (0) or on making a fortune (N).
- What is the probability of reaching N from k?

Assume that the other player (bank, casino, etc.) is capable of paying N bitcoins.

Introduction to Probability by Grinstead & Snell, American Mathematical Society (1997) Sec. 11.2 Exercises 23-25

Gambler's ruin

This is a homogeneous absorbing Markov chain with states $0, \ldots, N$:

- If the current fortune is m, the next bet can only lead to $m-1$ or $m+1$; i.e., outcome of the next bet is decided completely by the current fortune.
- Probabilities p and q do not change from bet to bet.
- The betting game ends on reaching a fortune of either 0 or N.

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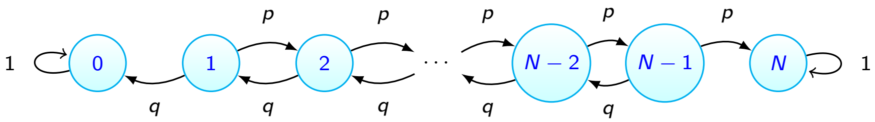
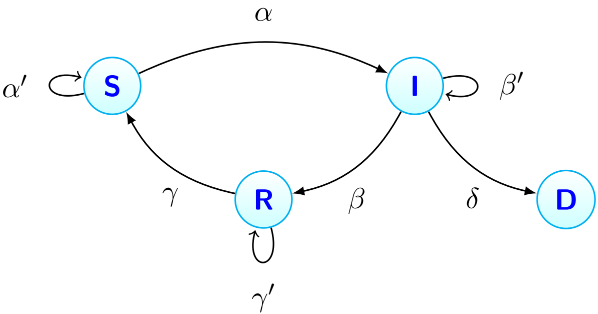
Gambler's ruin



$$\mathcal{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \\ N-2 \\ N-1 \\ N \end{array} \begin{array}{c} \begin{array}{ccccccc} 0 & 1 & 2 & \cdots & N-2 & N-1 & N \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ q & 0 & p & \cdots & 0 & 0 & 0 \\ 0 & q & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & p & 0 \\ 0 & 0 & 0 & \cdots & q & 0 & p \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Introduction to Probability by Grinstead & Snell, American Mathematical Society (1997) Sec. 11.2 Exercises 23-25

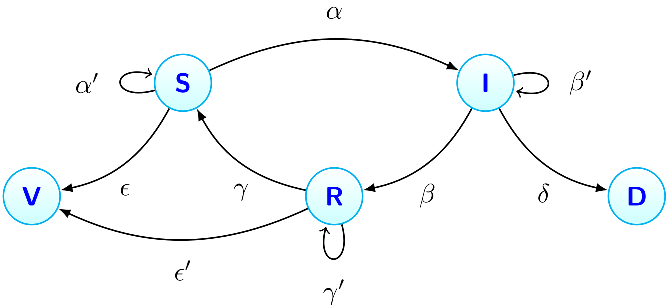
Army of the Dead



$$
\begin{array}{c|ccc|c|l}
 & \mathbf{S} & \mathbf{I} & \mathbf{R} & \mathbf{D} & \\
\mathbf{S} & \alpha' & \alpha & 0 & 0 & \alpha + \alpha' = 1 \\
\mathbf{I} & 0 & \beta' & \beta & \delta & \beta + \beta' + \delta = 1 \\
\mathbf{R} & \gamma & 0 & \gamma' & 0 & \gamma + \gamma' = 1 \\
\hline
\mathbf{D} & 0 & 0 & 0 & 1 &
\end{array}
$$

Example 4

Reason for Hope :: Vaccine providing lifelong immunity



$$
\begin{array}{c|ccc|cc}
 & \mathbf{S} & \mathbf{I} & \mathbf{R} & \mathbf{D} & \mathbf{V} \\
\mathbf{S} & \alpha' & \alpha & 0 & 0 & \epsilon \\
\mathbf{I} & 0 & \beta' & \beta & \delta & 0 \\
\mathbf{R} & \gamma & 0 & \gamma' & 0 & \epsilon' \\
\hline
\mathbf{D} & 0 & 0 & 0 & 1 & 0 \\
\mathbf{V} & 0 & 0 & 0 & 0 & 1
\end{array}
$$

$$\alpha + \alpha' + \epsilon = 1$$
$$\beta + \beta' + \delta = 1$$
$$\gamma + \gamma' + \epsilon' = 1$$

Definition. A state (say, the ith) of a Markov chain is called *absorbing* if it is impossible to leave it (i.e., $\mathcal{P}_{ii} = 1$).

Definition. A Markov chain is *absorbing* if it has at least one absorbing state, and if from every state it is possible to go to an absorbing state (not necessarily in one step).

Definition. In an absorbing Markov chain, a state which is not absorbing is called *transient*.

Introduction to Probability by Grinstead & Snell, American Mathematical Society (1997) Sec. 11.2

Rearrange states as { t TRansient states, r ABsorbing states }

$$\mathcal{P} = \begin{array}{c|c|l} \text{TR} & \text{AB} & \\ \hline \mathbf{Q} & \mathbf{R} & \text{TR} \\ \hline \mathbf{0} & \mathbf{I} & \text{AB} \end{array}$$

- **Q** :: $t \times t$ :: transitions between transient states
- **R** :: $t \times r$ :: transitions from transient to absorbing states
- **0** :: $r \times t$ :: all zeros – i.e., can't leave any absorbing state
- **I** :: $r \times r$ :: identity matrix

Show this!

$$\mathcal{P}^n = \begin{array}{c|c|c} \text{TR} & \text{AB} & \\ \hline \mathbf{Q}^n & \mathbf{S}_n\mathbf{R} & \text{TR} \\ \hline \mathbf{0} & \mathbf{I} & \text{AB} \end{array}$$

$$\mathbf{S}_n = \mathbf{I} + \mathbf{Q} + \ldots + \mathbf{Q}^{n-1}$$

Note

- $\mathbf{I}$ in the $\mathcal{P}^n$ matrix is an $r \times r$ identity matrix.
- $\mathbf{I}$ in the $\mathbf{S}_n$ matrix is an $t \times t$ identity matrix.

Theorem. For an absorbing Markov chain, $\mathbf{Q}^n \to 0$ as $n \to \infty$. That is, with probability 1, the markov chain gets absorbed.

This is Theorem 11.3 in *Introduction to Probability* by Grinstead & Snell, American Mathematical Society (1997).

Definition. The *Fundamental matrix* of an absorbing Markov chain is

$$\mathbf{N} = \lim_{n\to\infty} \mathbf{S}_n = \mathbf{I} + \mathbf{Q} + \mathbf{Q}^2 + \dots$$

Theorem. For an absorbing Markov chain,

1. The inverse of $\mathbf{N}$ is $\mathbf{I} - \mathbf{Q}$.
2. $\mathbf{N}_{ij}$ is the expected number of times the chain visits transient state j given that it started in transient state i.

This is Theorem 11.4 in *Introduction to Probability* by Grinstead & Snell, American Mathematical Society (1997).

Theorem. For an absorbing Markov chain starting in the *i*th transient state,

$$n_i = \sum_{j=1}^{t} \mathbf{N}_{ij}$$

is the expected number of steps (i.e., time) before being absorbed.

This is Theorem 11.5 in *Introduction to Probability* by Grinstead & Snell, American Mathematical Society (1997).

Theorem. For an absorbing Markov chain, the *ij*th element of the $t \times r$ matrix

$$\mathbf{B} = \mathbf{NR}$$

is the probability of the chain getting absorbed in the *j*th absorbing state starting from the *i*th transient state.

This is Theorem 11.6 in *Introduction to Probability* by Grinstead & Snell, American Mathematical Society (1997).

Gambler's ruin: Probability of winning a fortune

Gambler's ruin is an absorbing Markov chain with $0, N$ as the absorbing states and $1, \ldots, k, \ldots, N-1$ as the transients states.

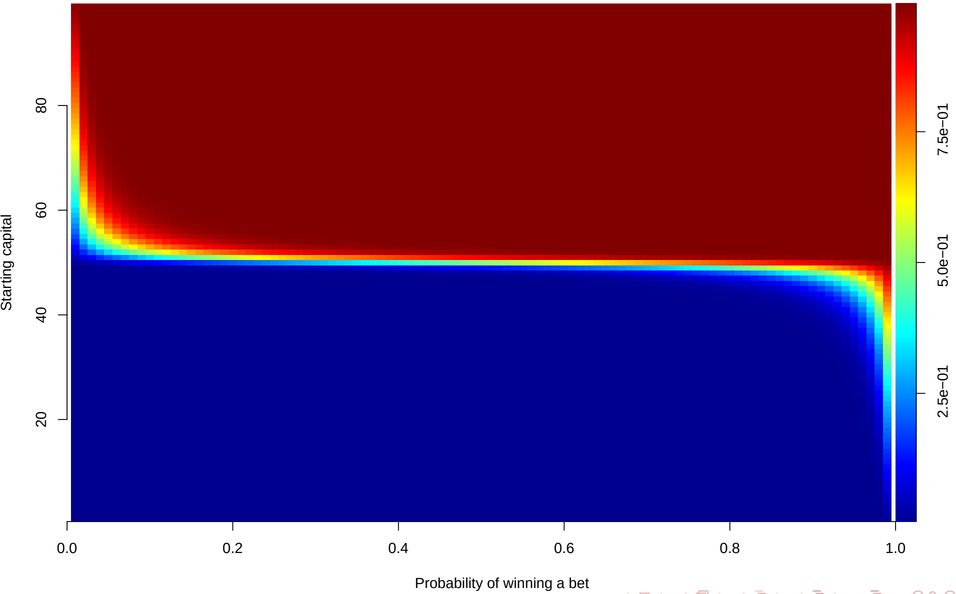
Proability of winning fortune N starting from initial fortune n

$$P(\text{reaching } N \text{ from } k) = \begin{cases} k/N & p = 1/2 \\ \frac{(q/p)^k - 1}{(q/p)^N - 1} & p \neq 1/2 \end{cases}$$

where p ($q = 1 - p$) is the probability of winning (losing) a bet.

Introduction to Probability by Grinstead & Snell, American Mathematical Society (1997) Sec. 11.2 Exercise 24

Gambler's ruin: Probability of winning a fortune

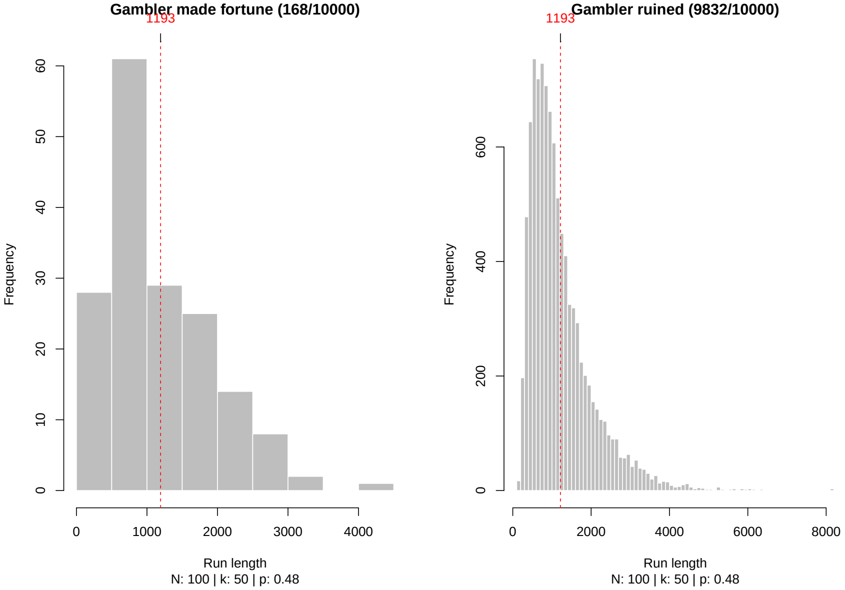


Simulating a Gambler's Ruin chain

```
gambling.run <- function( N, k, p )
 {
  sum <- 0; len <- 0
  while ( !( sum %in% c( -k, N - k ) ) ) # ruin and fortune respectively
   {
    sum <- sum + sample( c( +1, -1 ), 1, rep = TRUE, prob = c( p, 1 - p ) )
    len <- len + 1
   }

  c( sum, len )
 }

N <- 100   # gambler's target fortune
k <- 50    # gambler's initial capital
p <- 0.52  # gambler's P( winning one bet )
m <- 10000 # of replications / runs

stopifnot( length( N ) == 1, N > 0, N == as.integer( N ),
           length( k ) == 1, k > 0, k == as.integer( k ), k < N,
           length( p ) == 1, p > 0, p < 1,
           length( m ) == 1, m > 0, m == as.integer( m ) )

runs <- t( replicate( m, gambling.run( N, k, p ) ) ) # simulated runs

i <- which( runs[,1] == ( N - k ) ) # fortunes
j <- setdiff( 1:m, i )              # ruins

P.fortune <- length( i ) / m # estimated probability of winning fortune:
                             # compare number with the exact formula
```

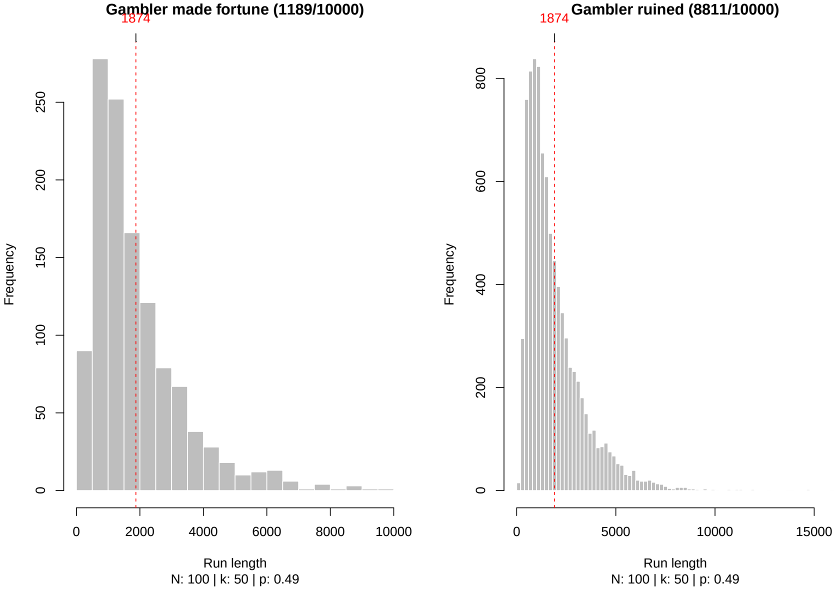
Gambler's ruin: Run length distributions

$p = 0.48, k = 50, N = 100$



Gambler's ruin: Run length distributions

$p = 0.49, k = 50, N = 100$



$p = 0.52, k = 50, N = 100$